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On central configurations of the κn-body problem

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Abstract

We consider planar central configurations of the Newtonian κn-body problem consisting in κ groups of regular n-gons of equal masses, called (κ, n)-crown. We derive the equations of central configurations for a general (κ, n)-crown. When $\kappa = 2$ we prove the existence of a twisted $(2, n)$-crown for any value of the mass ratio. Moreover, for $n = 3, 4$ and any value of the mass ratio, we give the exact number of twisted $(2, n)$-crowns, and describe their location. Finally, we conjecture that for any value of the mass ratio there exist exactly three $(2, n)$-crowns for $n \geq 5$.

1 Introduction

In the N-body problem a configuration is *central* if the acceleration vector for each body is a common scalar multiple of its position vector with respect to the center of mass. The study of central configurations allows to obtain explicit solutions of the N-body problem where the shape remains constant up to rescaling and rotation. While much is known about specific cases, usually involving symmetry or assuming that some bodies are infinitesimally small, less is known about the general structure of the set of central configurations. See Saari [9] for a introduction to the subject.

We focus on central configurations of the planar N-body problem for $N = \kappa n$, consisting in κ groups of n bodies located at the vertices of regular n-gons. In principle, no conditions on the masses of the same gon are imposed. Nevertheless, in the case of two regular n-gons, Zhang and Zhou [14] prove that the masses within each group must be equal. Although it is not known if that condition is necessary for more than two regular n-gons, we will restrict our study to central configurations such that all the bodies within the same gon have the same mass. We denote such central configurations by (k, n)-crowns (see Definition 1).

In Corbera et al. [3], the authors prove the existence of nested (κ, n)-crowns, where the bodies are at the vertices of κ homothetic regular n-gons (the vertices of the different n-gons are aligned), called nested n-gons, for all $\kappa \geq 2$ and $n \geq 2$. Zhao and Chen [15] prove the existence of central configurations of the $(pn + gn)$-body problem, where p regular n-gons are nested, and g regular n-gons are rotated exactly an angle π/n with respect the other ones. Llibre and Mello [5] show existence of (κ, n)-crowns for specific cases with $\kappa = 3, 4$.

In the case of two regular n-gons, Yu and Zhang [12] give a necessary condition for a $(2, n)$-crown: either the rings are nested or they must be rotated an angle π/n. Also, Yu and Zhang [13] wonder if the two regular n-gons can have different number of bodies, and the answer was negative. When the two gons are nested, Moeckel and Simó [7] prove that for every mass ratio, there are exactly two planar central configurations.

Beyond concentrating on the study of the existence of (k, n)-crowns for any given set of masses, our main aim is, as in Moeckel and Simó [7], to count how many there are.

The main goal of the paper is twofold. First, in Section 2, we present the general equations for central configurations of κ n-gons, each one with n bodies of the same mass. Each (κ, n)-crown is determined by three sequences, related to the masses, the radii of the κ circles where the different regular n-gons are inscribed, and the angles of rotation between the different κ n-gons. We also derive the equations for a (κ, n)-crown when all the angles of rotation are multiples of π/n, and we show two examples with $\kappa = 3$.

Second, in Section 3, we consider the case of $\kappa = 2$ *twisted* rings, where the two gons are rotated an angle π/n. In Theorem 1 we prove that for any value of the mass ratio and $n \geq 3$, there exists at least one $(2, n)$-crown. When $n = 3$ and $n = 4$, we give the exact number. More concretely, for $n = 3$, in Theorem 2, we show that the number varies between one and three; for $n = 4$, in Theorem 3, we show that for any value of the mass ratio there exists exactly three different crowns. Moreover, in both cases, $n = 3, 4$, we describe the set of admissible radii where the two twisted n-gons can be located. Finally, we conjecture that for any value of the mass ratio there exist exactly three $(2, n)$-crown for $n \geq 5$. Some results when $\kappa > 2$ will be presented in a forthcoming paper.

The paper include an Appendix where the detailed proof of some technical results are given.

2 Equations and definitions for general crowns

Consider the planar Newtonian N-body problem, $N = \kappa n$, consisting in κ groups of n bodies where all n bodies in the j-th group have equal mass m_j, $j = 1, \ldots, \kappa$. Let $\mathbf{q}_{ji} \in \mathbb{R}^2$, $j = 1, \ldots, \kappa$, $i = 1, \ldots, n$, be the position of each body in a reference frame where the center of mass is at the origin of coordinates. A *central configuration* of the κn-body problem is a configuration $\mathbf{q} = (\mathbf{q}_{11}, \mathbf{q}_{12}, \ldots, \mathbf{q}_{\kappa n}) \in \mathbb{R}^{2\kappa n}$ such that, for a value of $\lambda \in \mathbb{R}$, satisfies the equation

$$\nabla U(\mathbf{q}) + \lambda M \mathbf{q} = 0, \tag{1}$$

where U is the Newtonian potential

$$U(\mathbf{q}) = \sum_{j=1}^{\kappa} \sum_{i=1}^{n-1} \sum_{l=i+1}^{n} \frac{m_j^2}{||\mathbf{q}_{ji} - \mathbf{q}_{jl}||} + \sum_{j=1}^{\kappa-1} \sum_{l=j+1}^{\kappa} \sum_{i=1}^{n} \sum_{\nu=1}^{n} \frac{m_j m_l}{||\mathbf{q}_{ji} - \mathbf{q}_{l\nu}||},$$

and M is the diagonal matrix with diagonal $m_1, \ldots, m_1, \ldots, m_\kappa, \ldots, m_\kappa$ (each mass m_j repeated n times).

We are interested in central configurations such that all the bodies in the same group form a regular n-gon, also called *ring*.

Definition 1 *A central configuration formed by κ groups of n bodies in a regular n-gon such that all the masses of the same group are equal, is called a* crown of κ rings of n bodies, *or simply a (κ, n)–crown.*

We denote by $\mathbf{q}_j = \mathbf{q}_{j1}$ the position of the *leader* of each group, so once its position is known, all the others bodies in the same ring are fixed. Introducing polar coordinates, we can write

$$\mathbf{q}_j = a_j e^{\mathrm{i}\varpi_j}, \; \mathbf{q}_{ji} = \mathbf{q}_j e^{\mathrm{i}2\pi(i-1)/n}, \quad j = 1, \ldots, \kappa, \; i = 1, \ldots, n, \tag{2}$$

where $\varpi_j \in (-\pi/n, \pi/n]$ and $a_j > 0$ are the polar angle of the leader and the radius of the j-th ring, $j = 1, \ldots, \kappa$, respectively. Therefore, a (κ, n)-crown is determined by three sequences $(m_1, \ldots, m_\kappa)$, $(\varpi_1, \ldots, \varpi_\kappa)$ and $(a_1, \ldots, a_\kappa)$ of κ elements, with $m_j > 0$ and $a_j > 0$.

Proposition 1 *Consider a (κ, n)-crown with masses m_j, $j = 1, \ldots, \kappa$ and bodies located at $\mathbf{q}_{ji}$, $j = 1, \ldots, \kappa$, $i = 1, \ldots, n$ as in (2). Then, exists a constant λ such that the angles ϖ_j and the radii $a_j > 0$, $j = 1 \ldots, n$, must satisfy the set of equations*

$$\begin{aligned} &\frac{m_j}{a_j^2} \sum_{k=1}^{n-1} \frac{e^{\mathrm{i}2\pi k/n} - 1}{(2 - 2\cos(2k\pi/n))^{3/2}} + \\ &\sum_{\substack{l=1 \\ l \neq j}}^{\kappa} m_l \sum_{k=1}^{n} \frac{a_l e^{\mathrm{i}(\varpi_l - \varpi_j + 2\pi k/n)} - a_j}{(a_l^2 + a_j^2 - 2 a_l a_j \cos(\varpi_l - \varpi_j + 2\pi k/n))^{3/2}} + \lambda a_j = 0, \quad j = 1, \ldots, \kappa. \end{aligned} \tag{3}$$

Proof Due to the symmetries of the problem, only 2κ equations in (1) are independent. Thus, it is enough to satisfy the 2κ equations related to the leaders:

$$\frac{\partial U}{\partial \mathbf{q}_j} + \lambda m_j \mathbf{q}_j = 0, \quad j = 1, \ldots, \kappa.$$

Using (2), the set of equations given in (3) are obtained.

A first question arise: how many distinct (κ, n)-crowns exists for a given set of masses. That is, how many different sequences of radii and angles satisfy the system of equations (3) for a given set of masses. Clearly, the set of (κ, n)-crowns is invariant under rotations (around the origin) and dilations. In order to count the number of (κ, n)-crowns, we fix their size and identify the rotationally equivalent configurations. Therefore, we take $\varpi_1 = 0$ and $a_1 = 1$.

Definition 2 *For any fixed values of κ and n, consider a configuration of κ rings of n bodies as in (2). We say that $(1, a_2, \ldots, a_\kappa)$ and $(0, \varpi_2, \ldots, \varpi_\kappa)$, where $a_j > 0$ and $\varpi_j \in (-\pi/n, \pi/n]$, are* admissible if there exist a constant λ and a sequence of positive masses $(m_1, \ldots, m_\kappa)$ such that Equations (3) are satisfied.

The system of Equations (3) has $\kappa - 1$ degrees of freedom. It seems natural to fix the angles ϖ_j, $j = 2, \ldots, \kappa$, and look for admissible radii. A second question arise: is any sequence of angles admissible? In the case of $\kappa = 2$ rings, the answer is no. Yu and Zhang [12] show that there exist only two sequences of admissible angles (ϖ_1, ϖ_2): $(0, 0)$ and $(0, \pi/n)$. That is, the vertices of the two rings are aligned, or the vertices of the second one are located at the bisector lines of the vertices of the first ring.

In the case $\kappa \geq 3$, as far as we know, no necessary conditions have been given in terms of the angles ϖ_j, and all the examples known satisfy $|\varpi_l - \varpi_j| = 0, \pi/n$ for all l, j. For example, Llibre and Mello [5] show the existence of some $(3,3)$-crown and also some $(4,2)$-crown. Siluszyk [10] shows numerically the existence of some $(3,2)$-crowns where $a_2 = a_3$. In Figure 1 we show two examples of crowns of three rings (with a numerical accuracy up to 10^{-10}). In both cases, $\varpi_1 = \varpi_3 = 0$ and $\varpi_2 = \pi/n$. On the left, a $(3,3)$-crown with $m_1 = 1$, $m_2 = 3.38825460822497$ and $m_3 = 2.17072146363531$, $a_1 = 1$, $a_2 = 0.8$ and $a_3 = 0.44$. On the right, a $(3,4)$-crown with $m_1 = 1$, $m_2 = 32.46470791102244$, $m_3 = 1.074699197011822$, $a_1 = 1$, $a_2 = 2$, and $a_3 = 0.4$.

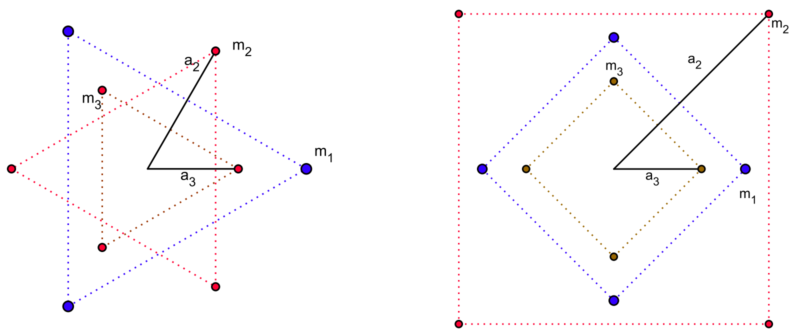

Figure 1: Two examples of twisted crowns. Left, a $(3,3)$-crown, with radii $a_3 < a_2 < a_1$. Right, a $(3,4)$-crown, with radii $a_3 < a_1 < a_2$. See the text for more details.

Notice that the sequence of admissible radii has no order established beforehand.

Lemma 1 *For any fixed value of κ and n, consider a sequence of $(\varpi_j)_{j=1}^n$ such that $|\varpi_l - \varpi_j| = 0, \pi/n$ for all l, j. Then*

$$\sum_{\substack{l=1 \\ l \neq j}}^{\kappa} m_l \sum_{k=1}^{n} \frac{a_l \sin(\varpi_l - \varpi_j + 2\pi k/n)}{(a_l^2 + a_j^2 - 2a_l a_j \cos(\varpi_l - \varpi_j + 2\pi k/n))^{3/2}} = 0, \quad j = 1, \ldots, \kappa, \tag{4}$$

for any value of m_j and $a_j > 0$, $j = 1, \ldots, \kappa$.

The proof is a straightforward calculation.

The imaginary part of the set of Equations (3) writes

$$\frac{m_j}{a_j^2}\sum_{k=1}^{n-1}\frac{\sin(2\pi k/n)}{(2-2\cos(2k\pi/n))^{3/2}}+\\ \sum_{\substack{l=1\\ l\neq j}}^{\kappa} m_l \sum_{k=1}^{n}\frac{a_l\sin(\varpi_l-\varpi_j+2\pi k/n)}{(a_l^2+a_j^2-2a_la_j\cos(\varpi_l-\varpi_j+2\pi k/n))^{3/2}}=0, \quad j=1,\ldots,\kappa.$$

Since the first sum is always zero, applying Lemma 1, when $|\varpi_l-\varpi_j|=0,\pi/n$ for all l,j, the second sum also vanishes and the set of Equations (3) reduce to κ equations with κ degrees of freedom. It is not known whether for other differences $|\varpi_l-\varpi_j|$ different from $0,\pi/n$ Equations (4) are satisfied for any value of m_j and a_j, $j=1,\ldots,\kappa$. Otherwise, when the sequence of angles follow specific relations or proportions, it is possible that some of the κ equations in (4) vanish and the total number of equations in (3) would also be reduced.

Proposition 2 *Consider a (κ,n)-crown with masses m_j, $j=1,\ldots,\kappa$ and bodies located at $\mathbf{q}_{ji}$, $j=1,\ldots,\kappa$, $i=1,\ldots,n$ as in (2), such that $|\varpi_l-\varpi_j|=0,\pi/n$ for all j,l. Then, exists a constant λ such that the radii $a_j>0$ must satisfy the set of equations*

$$-\frac{m_j}{a_j^2}S_n-\sum_{\substack{l=1\\ l\neq j}}^{\kappa} m_lC_{jl}(a_j,a_l)+\lambda a_j=0, \quad j=1,\ldots,\kappa, \tag{5}$$

where

$$S_n=\frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{\sin(k\pi/n)}, \tag{6}$$

and

$$C_{jl}=C_{jl}(a_j,a_l)=\sum_{k=1}^{n}\frac{a_j-a_l\cos(\varpi_j-\varpi_l+2k\pi/n)}{(a_j^2+a_l^2-2a_ja_l\cos(\varpi_j-\varpi_l+2k\pi/n))^{3/2}}, \tag{7}$$

for $j,l=1,\ldots,\kappa$

Proof Using $|\varpi_l-\varpi_j|=0,\pi/n$ for all j,l, then all κ equations in (4) are satisfied. And the real part of the system of equations (3) gives the system (5).

Under the hypothesis of Proposition 2, the system (5) has κ equations and 2κ unknowns: λ, the masses m_j for $j=1,\ldots,\kappa$ and the radii a_j, for $j=2,\ldots,\kappa$. We can eliminate λ using the equation of the first ring $j=1$, so we end up with the $\kappa-1$ following equations:

$$\left(C_{j1}-S_n\frac{a_j}{a_1^3}\right)m_1+\left(\frac{S_n}{a_j^2}-\frac{a_j}{a_1}C_{1j}\right)m_j+\sum_{\substack{l=2\\ l\neq j}}^{\kappa}\left(C_{jl}-\frac{a_j}{a_1}C_{1l}\right)m_l=0,\ j=2,\ldots,\kappa. \tag{8}$$

Furthermore, we can normalize the system (8) by setting $m_1=1$. Thus, system (8) have $\kappa-1$ equations with $2\kappa-2$ unknowns and writes:

$$(C_{j1}-S_na_j)+\left(\frac{S_n}{a_j^2}-a_jC_{1j}\right)m_j+\sum_{\substack{l=2\\ l\neq j}}^{\kappa}(C_{jl}-a_jC_{1l})\,m_l=0,\ j=2,\ldots,\kappa. \tag{9}$$

Definition 3 *Consider a (κ,n)-crown and let $\mathbf{q}_j=a_je^{\mathrm{i}\varpi_j}$, $a_j>0$, $j=1,\ldots,\kappa$, be the position of the leaders of each ring. We say that the j-th and k-th rings are* nested *if $\varpi_j-\varpi_k=0$, and are twisted if $|\varpi_j-\varpi_k|\neq 0$. A (κ,n)–crown with at least two twisted rings is called a twisted crown, whereas if all the rings are nested, is called a nested crown.*

From now on, we restrict our attention to the case of two twisted rings. We prove that for any set of masses there exists, at least, one $(2,n)$-crown. Furthermore, for a given set of masses we will count the number of $(2,n)$-crowns in the case of $n=3,4$. We will also give a conjecture for the general case $n\geq 5$.

3 Twisted crowns of two rings

We consider the case of twisted $(2,n)$-crowns, so that by Yu and Zhang [12], the only admissible angles are $\varpi_1 = 0$ and $\varpi_2 = \pi/n$. Then, the set of Equations (9) reduces to

$$C_2(a) - aS_n + \left(\frac{S_n}{a^2} - aC_1(a)\right) m = 0, \tag{10}$$

where

$$\begin{aligned} C_1(a) = C_{12}(1,a) &= \sum_{k=1}^{n} \frac{1 - a\cos((2k-1)\pi/n)}{(1 + a^2 - 2a\cos((2k-1)\pi/n))^{3/2}}, \\ C_2(a) = C_{21}(a,1) &= \sum_{k=1}^{n} \frac{a - \cos((2k-1)\pi/n)}{(1 + a^2 - 2a\cos((2k-1)\pi/n))^{3/2}}, \end{aligned} \tag{11}$$

and $a = a_2$, $m = m_2$. Equation (10) was also obtained by Roberts [8] and Yu and Zhang [12].

Solving Equation (10) with respect to the mass m as a function of the radius $a > 0$, we obtain the following expression:

$$m = H(a) = a\frac{F(a)}{F(1/a)}, \tag{12}$$

where

$$F(a) = S_n a - C_2(a). \tag{13}$$

We have used $C_1(a) = a^{-2}C_2(1/a)$ to derive the expression (12). Notice that $H(1) = 1$, which means that if the second ring is located on the same circle than the first one, then the masses of all bodies are equal, and the regular $2n$-gon central configuration is obtained.

The following result is straightforward.

Lemma 2 *Let H be the function defined in (12) for $a \in (0,\infty)$. Then $H(1/a) = 1/H(a)$ for any $a > 0$.*

Recall that, from Definition 2, an admissible sequence of radii is given by $(1,a)$. For a $(2,n)$-crown we define

$$\mathcal{A}_2(n) = \{a > 0; (1,a) \text{ is an admissible sequence for a } (2,n)\text{-crown}\}. \tag{14}$$

From Lemma 2 we have that if $a \in \mathcal{A}_2(n)$, so is $1/a$. That is, the $(2,n)$-crowns determined by the sequences $(1,m)$, $(1,a)$ and the sequences $(1,1/m)$, $(1,1/a)$ are qualitatively the same, in the sense that one is just the other one conveniently scaled.

The case of two twisted rings of two bodies ($n = 2$) is already known till 1932 from a work by MacMillan and Bartky [6] (see also Zhang and Zhou [14]): for any positive value of m, there exists only one central configuration. Moreover, the admissible values for the radius are $1/\sqrt{3} < a < \sqrt{3}$. Furthermore, it is not difficult to see that when $m > 1$ then $a < 1$, so that the bigger masses are always located in the inner ring, and the limit values $a = \sqrt{3}$ and $a = 1/\sqrt{3}$ correspond to the limit cases $m = 0$ and $m = \infty$, respectively.

Equations (10) and (12) are similar to the ones obtained by Moeckel and Simó [7] for the nested case $\varpi_1 = \varpi_2$, changing $\cos((2k-1)\pi/n)$ by $\cos(2k\pi/n)$. In the case of nested $(2,n)$–crowns, the authors prove that for any positive m there exist two central configurations. The proof is based in the fact that the function

$$\sum_{k=1}^{n} \frac{1}{(1 + a^2 - 2a\cos(k\pi/n))^{1/2}}$$

and all of its derivatives are positive. In our case the function involved is

$$\phi(a) = \sum_{k=1}^{n} \frac{1}{(1 + a^2 - 2a\cos((2k-1)\pi/n))^{1/2}}.$$

Then, function F, defined in (13) can be written as

$$F(a) = S_n a + \frac{d\phi}{da}(a).$$

While in the nested $(2,n)$–crowns, the function F is the sum of two increasing functions in the interval $(0,1)$, which implies that there exists only one root of $F(a)=0$ in that interval, in the twisted $(2,n)$–crowns, the function ϕ and its derivatives do not have constant monotonicity. So, we can not use the same arguments and we must adopt a different approach.

Next result can be found in [8] and [1]. Both works deal with the study of relative equilibria in the Maxwell's ring problem plus a central mass, where the same function F appears. Its proof is based on the fact that $F(1)=S_n-C_2(1)<0$, $\lim_{a\to 0^+}\frac{F(a)}{a}=S_n+\frac{n}{2}>0$, and $\lim_{a\to+\infty}F(a)=\lim_{a\to 0^+}F(1/a)=+\infty$.

Lemma 3 *Let F be the function defined in (13) for $n\geq 3$. Then, F is an analytic function for $a\in(0,\infty)$ and the equation $F(a)=0$ has at least two zeros $z_1<1<z_2$.*

The following Theorem prove the existence of, at least, one $(2,n)$-crown for any sequence of masses $(1,m)$.

Theorem 1 *For any natural number n and any value $m>0$ there exists a twisted $(2,n)$-crown with masses $m_1=1$ and $m_2=m$.*

Proof It is enough to prove that the function H defined in (12) has a range from 0 to ∞. Let be $z_1<z_2<\ldots<z_n$ all the roots of the equation $F(a)=0$. Therefore, they are all the zeros of $H(a)$ and $1/z_n<\ldots<1/z_1$ are all its poles. Recall that $H(1)=1$.

Let k be such that $z_k<1<z_{k+1}$. Suppose that $1/z_k<z_{k+1}$. Then, H is continuous in the interval $[z_k,1/z_k)$ and ranges from 0 to $+\infty$. Now suppose that $z_{k+1}<1/z_k$. Then, H is continuous and do not vanish in the interval $(1/z_{k+1},z_k]$, so it ranges from 0 to $+\infty$.

In the next subsections we give the exact number of twisted $(2,n)$-crowns for $n=3,4$ and for a fixed value of m. When $n\geq 5$ we give some partial results and present a Conjecture. Our methodology follows two steps: first, to study the set of admissible values $\mathcal{A}_2(n)$; second, to study the monotonicity of the function H.

3.1 Two twisted rings of three bodies

We consider a $(2,3)$-crown of two twisted triangles. In this case, the function F in (13) writes:

$$F(a) \;=\; \frac{\sqrt{3}}{3}\,a-\frac{2a-1}{(1+a^2-a)^{3/2}}-\frac{1}{(1+a)^2}. \qquad (15)$$

We start with a technical result that allows us to determine the set of admissible values of a for $n=3$. The proof can be found in the Appendix.

Lemma 4 *Consider the function $F(a)$ given in (15). Then $F(a)=0$ has exactly two positive solutions $z_1<1/2$ and $z_2>1$ satisfying $z_1z_2<1$.*

Proposition 3 *The set of admissible radii for twisted $(2,3)$-crowns is*

$$\mathcal{A}_2(3)=(0,z_1)\cup(1/z_2,z_2)\cup(1/z_1,\infty),$$

where z_1 and z_2 are given in Lemma 4.

Proof By definition of H in (12), its zeros are z_1 and z_2, and its poles are $1/z_2$ and $1/z_1$. From Lemma 4, $z_1z_2<1$ and therefore $0<z_1<1/z_2<z_2<1/z_1$. Furthermore, $F(0)=0$ and $F(1)<0$. Thus, $F(a)<0$ only for $a\in(z_1,z_2)$. Combining the signs of $F(a)$ and $F(1/a)$, the admissible radii $a>0$ for which $m=H(a)>0$ are

$$(0,z_1)\cup(1/z_2,z_2)\cup(1/z_1,\infty).$$

Next, in order to count the number of central configurations of two twisted equilateral triangles, we need information of the behavior of the function H.

Lemma 5 *Let H be the function given in (12) for $n = 3$ and $a > 1$. Then,*

1. *H has only two critical points: a local maximum at $1 < a < z_2$, and a local minimum at $1/z_1 < a < +\infty$, where z_1 and z_2 given in Lemma 4;*
2. *the equation $H(a) = a$ has only one positive solution at $a = 1$.*

The proof can be fount in the Appendix. Recall that the behavior of H for $a < 1$ can be recovered using Lemma 2.

Now, we can give the exact number of $(2, 3)$-crowns for any positive mass m.

Theorem 2 *Let be a central configuration of the 6-body problem corresponding to a twisted $(2, 3)$-crown with masses $m_1 = 1$, $m_2 = m$ and radii $a_1 = 1$, $a_2 = a$. Let z_1 and z_2 be given in Lemma 4. Then, there exist values $\overline{m}, \overline{M} > 1$ such that*

1. *for any $1 < m < \overline{m}$, there exists exactly three twisted $(2, 3)$-crowns. All of them with radius $1/z_2 < a < z_2$.*
2. *For any $\overline{m} < m < \overline{M}$ there exists only one twisted $(2, 3)$–crown with $1/z_2 < a < 1$.*
3. *For any $m > \overline{M}$, there exists exactly three twisted $(2, 3)$–crowns. One of them satisfies $1/z_2 < a < 1$ and the other two $a > 1/z_1$.*
4. *For $m = 1, \overline{m}, \overline{M}$ the number of different twisted $(2, 3)$–crowns is exactly two.*

Proof Let be $\overline{m}$ and $\overline{M}$ the values of H at the local maximum and minimum for $a > 1$ respectively, given by the first statement of Lemma 5. Using the second statement of the same Lemma, we have that $\overline{m} < \overline{M}$. The proof follows easily from Lemma 2 and the following properties of the function H:

1. $H(0) = H(z_1) = H(z_2) = 0$ and $H(1) = 1$;
2. $1/z_1$ and $1/z_2$ are poles of H.

Notice that using Lemma 2, the number of $(2, 3)$-crowns for $m < 1$ are obtained.

In Figure 2, the function H is plotted. The approximate bifurcation values are $\overline{m} = 1.0007682...$ and $\overline{M} = 35.70017694....$ In Figure 3 we show the three $(2, 3)$-crowns for $m = 40$.

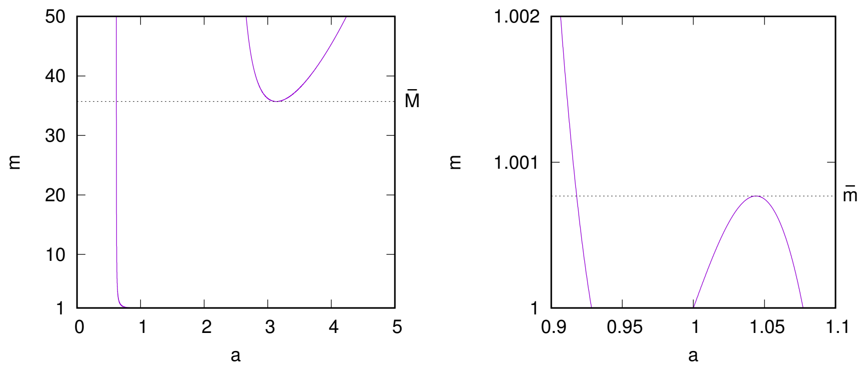

Figure 2: Graph of the function H for $n = 3$ and $m > 1$ (right: detail for $a \in [0.9, 1.1]$). The horizontal dotted lines correspond to the values $m = \overline{M}$ and $m = \overline{m}$ (see Theorem 2).

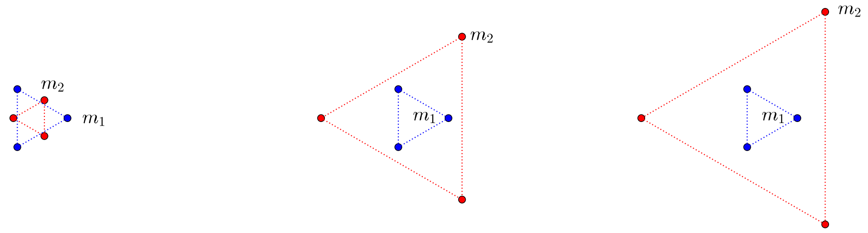

Figure 3: Example of the three $(2,3)$-crowns for $m_1 = 1$ and $m_2 = 40$. The radius of the first ring is $a_1 = 1$ while the radius of the second ring is $a_2 = 0.6190856888$, $a_2 = 2.810336016$ and $a_2 = 3.665961553$ respectively

3.2 Two twisted rings of four bodies

We consider $(2,4)$-crowns of two twisted squares. In this case, the function F in (13) writes:

$$F(a) \quad = \quad \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right) a - \frac{2a - \sqrt{2}}{(1 + a^2 - \sqrt{2}a)^{3/2}} - \frac{2a + \sqrt{2}}{(1 + a^2 + \sqrt{2}a)^{3/2}}. \tag{16}$$

Similarly to the previous subsection, we start with a technical result about the zeros of the function F (proof in the Appendix).

Lemma 6 *Consider the function $F(a)$ given in (16). Then $F(a) = 0$ has exactly two positive solutions $z_1 < 1 < z_2$ satisfying $z_1 z_2 > 1$.*

Notice that, the number of zeros of F is the same as in the case $n = 3$, but their product in this case is bigger than one. That is an important property to determine the admissible values of a for a (2,4)-crown.

Proposition 4 *The set of admissible radii for twisted $(2,4)$-crowns is*

$$\mathcal{A}_2(4) = (0, 1/z_2) \cup (z_1, 1/z_1) \cup (z_2, \infty),$$

where z_1 and z_2 are given in Lemma 6.

Proof By definition of H in (12), its zeros are z_1 and z_2, and its poles are $1/z_2$ and $1/z_1$. From Lemma 6, we have that $0 < 1/z_2 < z_1 < 1 < 1/z_1 < z_2$. Furthermore, $F(0) = 0$ and $F(1) < 0$. Thus, $F(a) < 0$ only for $a \in (z_1, z_2)$. Combining the signs of $F(a)$ and $F(1/a)$, the admissible values $a > 0$ for which $H(a) > 0$ are

$$(0, 1/z_2) \cup (z_1, 1/z_1) \cup (z_2, \infty).$$

Lemma 7 *Let H be the function given in (12) for $n = 4$. Then, H is monotone increasing.*

Finally we can establish the number of central configurations of twisted $(2,4)$-crowns.

Theorem 3 *Let be a central configuration of the 8-body problem corresponding to a twisted $(2,4)$-crown with masses $m_1 = 1$, $m_2 = m$ and radii $a_1 = 1$, $a_2 = a$. Let z_1 and z_2 be given in Lemma 6.*

1. *For any $m > 1$, there exist exactly three twisted $(2,4)$-crowns. One has radius $a < 1/z_2$, another one with $z_1 < a < 1/z_1$ and the last one with $a > z_2$.*

2. *For $m = 1$ there exist exactly two central configurations of twisted $(2,4)$-crown with radii $a = 1$ and $a > z_2$.*

Proof The proof is straightforward taking into account that z_1, z_2 are the zeros of H, $1/z_1, 1/z_2$ its poles, and that H is monotone increasing, so any equation $m = H(a)$ has three solutions, each one in one of the intervals of the admissible set $\mathcal{A}_2(4)$. In the case $m = 1$, if $a \neq 1$ is a solution of $H(a) = 1$, the crowns with radii a and $1/a$ represent the same twisted crown rescaled.

3.3 Two twisted rings of $n \geq 5$ bodies

The main obstruction to give the exact number of twisted $(2, n)$-crowns is to determine the number of solutions of $F(a) = 0$, where F is defined in (13). Several authors have deal with that problem for a general value n, but as far as we know, no one has been able to prove that F has exactly two zeros.

It is well known that F (see Lemma 3) has at least two zeros for any $n \geq 3$, denoted by z_1 and z_2. Some results can be given supposing that z_1 and z_2 are the only positive solutions of $F(a) = 0$. Roberts, in [8], shows that for $n \geq 5$

$$0 < 1 - \frac{1}{n} < z_1 < 1 < \frac{1}{z_1} < \frac{n}{n-1} < z_2.$$

Using the results recently obtained in Barrabés and Cors [2], we can give a better bound.

Lemma 8 *Let be $n \geq 5$ and suppose that z_1 and z_2 are the only positive solutions of $F(a) = 0$, where F is given in (13). Then*

$$0 < \cos\left(\frac{\pi}{n}\right) < z_1 < 1 < \frac{1}{z_1} < \frac{1}{\cos\left(\frac{\pi}{n}\right)} < z_2.$$

Proof The results follows from the fact that for any $n \geq 5$, $F(a_n) > 0$ and $F(1/a_n) < 0$, where $a_n = \cos\left(\frac{\pi}{n}\right)$, as is stated in Barrabés and Cors [2], and the fact that $F(1) < 0$, proved in [8].

In Table 1 we show the approximate numeric values of z_1 and z_2 for some values of n.

n	z_1	z_2
3	0.413887932417	1.619789608802
4	0.697380509876	1.602408486212
5	0.822182869908	1.597921728909
6	0.884321138125	1.592235355387
7	0.918990363772	1.584120901279
8	0.940138179122	1.574515176634
9	0.953949939513	1.564321826382
10	0.963459881269	1.554123467683
$\vdots$	$\vdots$	$\vdots$
100	0.999674025507	1.352557858581
500	0.999986989988	1.279569044474
1000	0.999996754292	1.256683821749
5000	0.999999869916	1.215703126473

Table 1: Values of z_1 and z_2 for the values of n shown.

Conjecture 1 *For any $n \geq 5$:*

1. *The function F, defined in (13), has only two positive solutions, denoted by z_1 and z_2.*
2. *The set of admissible values for a $(2, n)$-crown is*

$$\mathcal{A}_2(n) = (0, 1/z_2) \cup (z_1, 1/z_1) \cup (z_2, \infty).$$

3. *For any value of the mass ratio $m > 1$, there exists exactly three twisted $(2, n)$-crowns.*

4. *For $m = 1$, there exists exactly two twisted $(2, n)$-crowns.*

Notice that item 2 of the above Conjecture is an immediate consequence of the first one and Lemma 8. If z_1 and z_2 are the only zeros of F, we have that $z_1 \cdot z_2 > 1$ and the set of admissible values $\mathcal{A}_2(n)$ is obtained. Then, using that z_i and $1/z_i$ are zeros and poles (respectively) of the function $m = H(a)$, its range is $[0, \infty)$ in each one of the disjoint intervals of $\mathcal{A}_2(n)$. Then, there exist, at least, three twisted $(2, n)$-crowns. For $n \leq 100$, we have checked numerically also that H is monotonic increasing, so the number of crowns is exactly three, except for $m = 1$, which is two. In Figure 4 we plot the function $H(a)$ for several values of n.

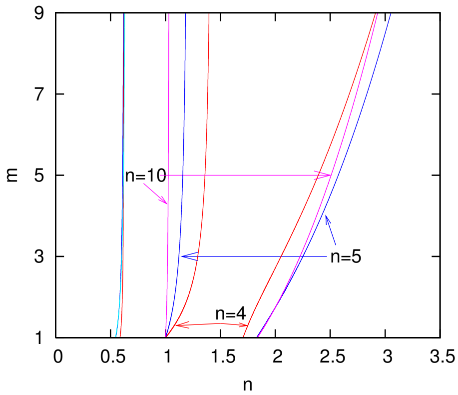

Figure 4: The curves $m = H(a)$ for $n = 4, 5, 10$.

4 Acknowledgements

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5 Appendix

In this section we give proof of some of the Lemmas used. In some cases, one can convince himself of the certainty of the results simply by plotting the graphic of a function. Nevertheless, we give here a rigorous analytical proof of each one of them.

All the results to prove involve the problem of findings roots of an equation. To tackle the issue, we make use of two classic tools. On one hand, we will utilize Sturm's theorem to give a rigorous computer-algebra assisted proof in the case of zeros of polynomials with integer coefficients. On another hand, we will apply Theorem 2 of Voorhoeve and Van Der Poorten [11] in order to have un upper bound of the total number of zeros of a function in an interval. We recall here that result.

Let $P(a) = \sum_{k=1}^{m} p_k(a)$, where p_k are real analytic functions in an interval $I = [a_0, a_1] \subset \mathbb{R}$. Then the number of zeros of P in the interval I counted according to multiplicity, $N_I(P)$, satisfies

$$N_I(P) \leq m - 1 + \sum_{k=1}^{m} N_I(W(p_1, \ldots, p_k)) + \sum_{k=1}^{m-2} N_I(W(p_1, \ldots, p_k)), \tag{17}$$

where $W(p_1, \ldots, p_k)$ denotes the Wronskian determinant of the functions involved.

Finally, we also study the monotonicity of the function H. From its expression given in (12), we have that

$$H'(a) = \frac{aF(a)F(1/a) + a^2F'(a)F(1/a) + F(a)F'(1/a)}{aF(1/a)^2}. \tag{18}$$

The denominator of (18) is always positive, so we will study the sign of its numerator.

5.1 Proof of Lemma 4

We want to prove that the equation $F(a) = 0$, where F is given in (15):

$$F(a) = \frac{\sqrt{3}}{3}\,a - \frac{2a-1}{(1+a^2-a)^{3/2}} - \frac{1}{(1+a)^2},$$

has exactly two positive solutions $z_1 < 1/2$ and $z_2 > 1$ satisfying $z_1z_2 < 1$.

We notice that $F(1/2) \neq 0$. Then, the solutions of $F(a) = 0$ are the same ones of $\overline{F}(a) = 0$, where

$$\overline{F}(a) = \underbrace{\frac{-1}{(1+a^2-a)^{3/2}}}_{f_1} + \underbrace{\frac{1}{3}\frac{\sqrt{3}a^3 + 2\sqrt{3}a^2 + \sqrt{3}a - 3}{(1+a)^2(2a-1)}}_{f_2} = f_1(a) + f_2(a),$$

and the functions f_1 and f_2 are real analytic in the intervals $I_1 = [\epsilon, 1/2 - \epsilon]$ and $I_2 = [1/2 + \epsilon, K]$ for any $\epsilon > 0$ and $K >> 1/2$. We study the number of solutions of $\overline{F}(a) = 0$ in each interval.

The following limits

$$\begin{array}{rclrcl}
\lim\limits_{a\to(1/2)^-} \overline{F}(a) & = & +\infty, & \lim\limits_{a\to(1/2)^+} \overline{F}(a) & = & -\infty, \\
\lim\limits_{a\to 0^+} \dfrac{\overline{F}(a)}{a} & < & 0, & \lim\limits_{a\to+\infty} \overline{F}(a) & = & +\infty,
\end{array}$$

imply that the function $\overline{F}$ has at least one zero in each interval I_j, $j = 1, 2$ for ϵ small enough and K big enough, and the total number of zeros must be odd. Applying (17), and using that f_1 has no zeros, we have that

$$1 \le N_{I_j}(\overline{F}(a)) \le 1 + N_{I_j}(W(f_1, f_2)), \qquad j = 1, 2, \tag{19}$$

where

$$W(f_1, f_2) = \frac{-1}{6}\frac{\Delta(a)}{(1+a^2-a)^{5/2}(1+a)^3(2a-1)^2}$$

and

$$\Delta(a) = 12\sqrt{3}a^6 + 22\sqrt{3}a^5 - \sqrt{3}a^4 - 17\sqrt{3}a^3 - (5\sqrt{3}+36)a^2 + (63-\sqrt{3})a - 9 - 2\sqrt{3}.$$

We first examine the number of roots of $\Delta(a)$ in the interval I_2. Introducing a translation, we can write

$$\Delta(b + 1/2) \ge \frac{3\sqrt{3}}{8}\,b\,(176b^2 + (18 - 32\sqrt{3})b + 24\sqrt{3} - 27) > 0 \qquad \text{for} \quad b > 0.$$

Therefore $\Delta(a)$ has no zeros in I_2, and, using (19), $N_{I_2}(\overline{F}(a)) = 1$.

Next, we study the number of roots of $\Delta(a)$ in the interval I_1. On one hand we have that $\Delta(0)\cdot\Delta(1/2) < 0$, so at least there exists one zero. On the other hand, using that $a \in (0, 1/2)$,

$$\frac{\Delta'(a)}{\sqrt{3}} > 110a^4 - 55a^2 + 9 > 0.$$

Then, $\Delta(a) = 0$ has exactly one solution in the interval I_1, so that by (19)

$$1 \le N_{I_1}(\overline{F}(a)) \le 2.$$

But the total number of zeros of $\overline{F}$ in I_1 must be odd because of the change of sign of $\overline{F}$ in the interval I_1. Therefore, $N_{I_1}(\overline{F}(a)) = 1$.

Finally, we know that $z_1 < 1/2$ and it is easy to see that $F(1) < 0$ and $F(2) > 0$, so $1 < z_2 < 2$. Therefore, $z_1z_2 < 1$.

5.2 Proof of Lemma 5

Let F be the function given in (15), and let H be defined as in (12). We look for the critical points of H. We introduce the expression of F in the expression for H' (18), and multiply it by $2\sqrt{3}(1+a^2-a)^{5/2}(1+a)^4$. Then, the critical points of H are also solutions of the equation

$$E(a) = p_1(a) + p_2(a)\sqrt{1+a^2-a} = 0,$$

where

$$\begin{aligned} p_1(a) &= 4\,a^{10} - a^9 - 23\,a^8 + \left(-23 - 2\sqrt{3}\right)a^7 + \left(-1 - 17\sqrt{3}\right)a^6 \\ &\quad + \left(8 + 42\sqrt{3}\right)a^5 + \left(-1 - 17\sqrt{3}\right)a^4 + \left(-23 - 2\sqrt{3}\right)a^3 - 23\,a^2 - a + 4, \\ p_2(a) &= -4\,a^{10} + \left(4 + 2\sqrt{3}\right)a^9 + \left(-4 + 4\sqrt{3}\right)a^8 + \left(-4 + 4\sqrt{3}\right)a^7 + \left(24\sqrt{3} + 4\right)a^6 \\ &\quad + \left(26\sqrt{3} - 8\right)a^5 + \left(24\sqrt{3} + 4\right)a^4 + \left(4\sqrt{3} - 4\right)a^3 + \left(4\sqrt{3} - 4\right)a^2 + \left(4 + 2\sqrt{3}\right)a - 4. \end{aligned}$$

It is easy to check that $E(1) > 0$, $E(2) < 0$ and $E(4) > 0$, so at least there exist two roots of $H' = 0$ located in $I_1 = [1,2]$ and $I_2 = [2,4]$. Thus, $N_{I_j}(H') \geq 1$, $j = 1,2$. We also consider the interval $I_3 = [4,\infty)$. Therefore, applying (17), the number of roots (counting multiplicity) of $H' = 0$ in each interval is bounded by

$$N_{I_j}(H') \leq 1 + N_{I_j}(p_1) + N_{I_j}(W(p_1, f_2)), \quad j = 1,2,3,$$

where $f_2(a) = p_2(a)\sqrt{1+a^2-a}$.

Using Descarte's rule (after a suitable translation if needed), it is not difficult to see the following properties:

- p_1 has only one zero for $a > 1$ located in I_2.
- $\sqrt{1+a^2-a}\,W(p_1, f_2) = (a-1)p_3(a)$, where $p_3(a)$ is a polynomial of degree 20 with only one zero for $a > 1$ located at I_3.

Then, it follows that $N_{I_j}(H') = 1$, for $j = 1,2$. Finally, to see that there are no roots in $[4,\infty)$, it is enough to see that $p_1^2(a) - (1+a^2-a)p_2^2(a) > 0$ in that interval.

So far, we have seen that there exist a critical point of H in $[1,2]$ and another one in $[2,4]$. To finish the proof of the first statement, notice that $H(1) = 1$ and $H(z_2) = 0$ and z_1 is a pole of H. Thus, the first critical point must be a local maximum at $(1, z_2)$ and the second one must be a local minimum at $(1/z_1, \infty)$. See Figure 2.

The second statement claims that the only positive solution of $H(a) = a$ is $a = 1$. The equation can be written as

$$\frac{\sqrt{3}}{3}(a^2-1) = \frac{a(a-1)(a^2-a+1)}{(a^2-a+1)^{3/2}} + \frac{a(1-a^2)}{(1+a)^2}.$$

Clearly the equation has $a = 1$ as a solution. Simplifying by the term $a - 1$ at both sides and rearranging we obtain

$$\frac{\sqrt{3}}{3}(a+1) + \frac{a}{(1+a)} = \frac{a}{(a^2-a+1)^{1/2}}.$$

The study of the derivative of the functions of both sides of the equations leads quickly to the conclusion that the equation has no positive solutions. Therefore, $a = 1$ is the only positive solution of $H(a) = 0$. This concludes the proof.

5.3 Proof of Lemma 6

We want to proof that the function $F(a)$ given in (16):

$$F(a) = \left(\frac{1}{4} + \frac{\sqrt{2}}{2}\right)a - \frac{2a - \sqrt{2}}{(1+a^2-\sqrt{2}a)^{3/2}} - \frac{2a + \sqrt{2}}{(1+a^2+\sqrt{2}a)^{3/2}},$$

has exactly two positive roots z_1 and z_2 satisfying $z_1 z_2 > 1$.

We introduce the change $a = b\sqrt{2}/2$ and the function writes

$$F(b) \quad = \quad \frac{\sqrt{2}+4}{8}b - \frac{4(b-1)}{(b^2-2b+2)^{3/2}} - \frac{4(b+1)}{(b^2+2b+2)^{3/2}}$$

It is not difficult to see that $F(\sqrt{2}) < 0$,

$$\lim_{b\to 0} \frac{F(b)}{b} > 0, \qquad \text{and} \qquad \lim_{b\to+\infty} F(b) = +\infty.$$

Therefore, there exist at least one root in $(0, \sqrt{2})$, and in $(\sqrt{2}, \infty)$ (respectively in the intervals $(0, 1)$ and $(1, \infty)$ in the variable a).

Let $\overline{F}$ be

$$\begin{aligned} \overline{F}(b) &= \frac{(b^4+4)^{3/2}}{4} F(b) = \underbrace{\frac{\sqrt{2}+4}{32} b(b^4+4)^{3/2}}_{f_1} \\ &\quad + \underbrace{\left(-(b-1)(b^2+2b+2)^{3/2} - (b+1)(b^2-2b+2)^{3/2}\right)}_{f_2} \\ &= f_1(b) + f_2(b) \end{aligned}$$

Consider the intervals $I_1 = [\epsilon, \sqrt{2}]$ and $I_2 = [\sqrt{2}, M]$, for $\epsilon > 0$ small enough and $M >> \sqrt{2}$ big enough. We know that there exists already one root of $\overline{F}(b) = 0$ in each interval. Using that $f_1(b) \neq 0$ for $b > 0$ and applying (17), we have that

$$1 \le N_{I_j}(\overline{F}(b)) \le 1 + N_{I_j}(W(f_1, f_2)), \qquad j = 1, 2.$$

where

$$W(f_1, f_2) = \frac{\sqrt{2}+4}{32}(b^4+4)^{1/2}\left(p_1(b)(b^2-2b+2)^{5/2} + p_2(b)(b^2+2b+2)^{5/2}\right),$$

and $p_1(b) = 3b^3 + 7b^2 + 5b + 2$ and $p_2(b) = 3b^3 - 7b^2 + 5b - 2$. On one hand, $p_1(b) > 0$ for $b > 0$ and $p_2(b) > 0$ only for $b > b_0 \simeq 1.528181327...$ (the only real root of p_2). Then, $W(f_1, f_2)$ has no zeros for $b \ge b_0$. On the other hand, the roots of $W(f_1, f_2) = 0$ will be also roots of

$$p_1(b)^2(b^2-2b+2)^5 - p_2(b)^2(b^2+2b+2)^5 = 0.$$

The equation corresponds to a polynomial of degree 12 with only two real roots. By a Bolzano argument, only one of the roots is smaller that b_0 and could be a zero of $W(f_1, f_2)$. Therefore, we have that $N_{I_j}(W(f_1, f_2)) \le 1$, $j = 1, 2$. Using the fact that there is a change of sign in each interval we conclude that

$$N_{I_j}(F) = 1, \qquad j = 1, 2.$$

To finish the proof of the Lemma, using a Bolzano's argument we have that $\frac{13}{20} < z_1 < \frac{7}{10}$ and $\frac{8}{5} < z_1 < \frac{33}{20}$. Therefore, $z_1 z_2 > 1$.

5.4 Proof of Lemma 7

We want to prove that the function $H(a)$ given in (12) for $n = 4$ is monotonic increasing. Recall that from Lemma 2, it is enough to show that $H'(a) > 0$ for $a \in (0, 1]$, or equivalently, that the numerator of (18) is positive. Multiplying that expression by x^3y^3, where

$$x = \sqrt{1 + a^2 - \sqrt{2}a}, \qquad y = \sqrt{1 + a^2 + \sqrt{2}a},$$

and simplifying, we get that $H'(a) > 0$ is equivalent to

$$Q_0 + Q_1 x + Q_2 y + Q_3 xy > 0, \tag{20}$$

where

$$
\begin{aligned}
Q_0 &= 256\left(4\sqrt{2}-9\right)\left(a^2-a-1\right)\left(a^2+a-1\right)a^4,\\
Q_1 &= 14\left(\sqrt{2}-4\right)\left(a^4-a^3+a^2-a+1\right)\left(4a^2+7\sqrt{2}a+4\right)(a+1)\left(1+a^2-\sqrt{2}a\right)^2,\\
Q_2 &= -14\left(\sqrt{2}-4\right)\left(a^4-a^3+a^2-a+1\right)\left(4a^2-7\sqrt{2}a+4\right)(a+1)\left(1+a^2+\sqrt{2}a\right)^2,\\
Q_3 &= \left(147\,a^8+\left(2304-1024\sqrt{2}\right)a^5+294\,a^4+\left(2304-1024\sqrt{2}\right)a^3+147\right)a.
\end{aligned}
$$

Assume for a moment that for $a \in (0,1]$

$$
Q_2+Q_3x>0, \quad \text{and} \quad R_0+R_1x>0, \tag{21}
$$

where $R_0 = Q_0+Q_2$ and $R_1 = Q_1+Q_3$. Then, using these inequalities and the fact than $y>1$, the expression (20) satisfies

$$
Q_0+Q_1x+Q_2y+Q_3xy = Q_0+Q_1x+(Q_2+Q_3x)y > R_0+R_1x>0,
$$

and the proof of the Lemma is finished.

It remains to prove the two claims (21). We use a rational parametrization (see, for example, [4], where the authors apply the methodology to some problems of central configurations) given by the change

$$
a\sqrt{2}-1 = \frac{t^2-1}{2t}, \qquad t \in I = \left(\sqrt{2}-1, \sqrt{2}-1+\sqrt{4-s\sqrt{2}}\right],
$$

(recall $a\in(0,1]$). Then,

$$
\begin{aligned}
Q_2+Q_3x &= \frac{2\sqrt{2}-1}{32768\,t^{11}}p(t), &(22)\\
R_0+R_1x &= \frac{\sqrt{2}-4}{65536\,t^{12}}(t+1-\sqrt{2})q(t), &(23)
\end{aligned}
$$

where $p(t)$ and $q(t)$ are polynomials of degree 22 and 23, respectively.

In (22), we bound from below $p(t)$ by deleting the terms of order bigger than 19, and replacing the coefficients of the other orders a_i by $\lfloor a_i \rfloor$ (the floor function):

$$
\begin{aligned}
p(t) > {} & 63196\,t^{18}-126225\,t^{17}-624027\,t^{16}+61908\,t^{15}+3070770\,t^{14}+1765978\,t^{13}\\
&-6817833\,t^{12}-1430685\,t^{11}+5388132\,t^{10}-211516\,t^{9}+2666042\,t^{8}-4291425\,t^{7}+\\
&2330295\,t^{6}-590575\,t^{5}+45780\,t^{4}+13874\,t^{3}-4517\,t^{2}+563\,t-28.
\end{aligned}
$$

The later polynomial does not have zeros in the interval $\overline{I}$ applying Sturm's theorem and it is positive for $t\in I$. Thus, $p(t)>0$ for $t \in I$.

In (23), the multiplying constant is negative, so we want to see that $q(t)<0$ for $t\in I$. In this case we bound from above $q(t)$ by replacing its coefficients a_i by $\lceil a_i \rceil$ (the ceil function).

$$
\begin{aligned}
q(t) < {} & 14\,t^{23}+261\,t^{22}+1520\,t^{21}+775\,t^{20}-16986\,t^{19}+42195\,t^{18}+725386\,t^{17}+1605760\,t^{16}\\
&-3234196\,t^{15}-15882376\,t^{14}-9672761\,t^{13}+28027022\,t^{12}+20951111\,t^{11}-20101757\,t^{10}\\
&-8955856\,t^{9}+3502769\,t^{8}+3324573\,t^{7}-2004050\,t^{6}+323005\,t^{5}+38192\,t^{4}\\
&-26449\,t^{3}+6058\,t^{2}-723\,t+34.
\end{aligned}
$$

Again, the later does not have zeros in the interval $\overline{I}$ applying Sturm's theorem and it is negative for $t\in I$. Therefore, $q(t)>0$ for $t\in I$.

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